Orthogonality relation for a three-dimensional scattering electromagnetic field in a dispersive medium

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Considering a scattering field by a perfectly conducting sphere as a typical example, we develop a method to derive an orthogonality relation for a three-dimensional scattering electromagnetic field in a dispersive dielectric medium. Each orthogonal mode is composed of the doublet of the incident plane field and the scattering spherical plane field. The scattering field includes the near field in the vicinity of the scatterer. Expanding the total field energy stored in the whole space using the derived orthogonality relation, we show that the total field energy is expressed as the sum of the energies of independent harmonic oscillators.

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I. INTRODUCTION

The orthonormal set of plane electromagnetic waves in a homogeneous nondispersive medium, such as a vacuum, is well known for providing a basis for obtaining the photon picture of electromagnetic waves under cyclic boundary conditions on the wall of a medium with cubic spatial volume. Here, the field energy is expressed as the sum of the energies of independent harmonic oscillators with the use of the orthogonal set. The photon picture of the electromagnetic field in a dispersive medium has also been developed in infinite transparent dielectrics [1], in nonlinear and inhomogeneous transparent dielectrics under the approximation of slowly varying amplitude [2], and in dielectric media that exhibit both loss and dispersion in an infinite medium [3]. However, if we are concerned with evanescent waves having a pure imaginary wave vector in the direction of propagation, we cannot apply the above method to obtain the orthonormal set. Concerning this, a pioneering work has been performed on the quantization of the evanescent waves appearing as a result of the total reflection at a plane boundary [4]. Here it was shown that the triplet of the incident, reflected, and evanescent transmitted plane waves composed an orthonormal set. Expanding the above concept, we have also derived an extended orthonormal relation for the evanescent electromagnetic fields in a dispersive medium with a dielectric constant $\varepsilon(\omega)$ in which the dispersive property $\partial\varepsilon(\omega)/\partial\omega$ of the dielectric medium was taken into account [5].

In the present paper, we further develop a generalized method by which to derive an orthonormal set for complex scattering fields in a dispersive medium; considering a scattering field by a perfectly conducting sphere immersed in a dispersive medium as a typical illustration, we demonstrate a method to derive the orthonormal set. Here, each orthonormal set is composed of the doublet of the incident plane wave and its scattering spherical one. The near field in the vicinity of the scatterer is also taken into account in the scattering field. By using the derived relation, we expand the total field energy stored in the whole space and show that the total field energy is expressed as the sum of the energies of independent harmonic oscillators.

II. SPHERICAL MODE FUNCTIONS OF THE ELECTROMAGNETIC FIELD

Let us consider the scattering problem shown in Fig. 1, where a perfectly conducting sphere with radius a is embedded in a homogeneous, lossless, dispersive dielectric medium. The dielectric constant ε of the medium thus depends on the angular frequency ω as

$$\varepsilon=\varepsilon(\omega)\geqslant\varepsilon_0. \tag{1}$$

A plane electromagnetic wave is assumed to propagate with a wave vector $\mathbf{k}=k(\sin\theta_0\cos\phi_0,\sin\theta_0\sin\phi_0,\cos\theta_0)$, where k is always a positive real number, θ_0, ϕ_0 are the direction angles of $\mathbf{k}$, and

$$k^2=|\mathbf{k}|^2=\omega^2\varepsilon\mu_0. \tag{2}$$

The plane wave incident on the perfectly conducting sphere is scattering. Therefore, the total electromagnetic field in the whole space is composed of the incident field and the scattering one. The total electric field $\mathbf{e}(\mathbf{k},\mathbf{r})$ and the total magnetic field $\mathbf{b}(\mathbf{k},\mathbf{r})$ each include the incident field labeled by the superscript i and the scattering field labeled by the superscript s,

$$\begin{aligned}\mathbf{e}(\mathbf{k},\mathbf{r})&=\mathbf{e}^i(\mathbf{k},\mathbf{r})+\mathbf{e}^s(\mathbf{k},\mathbf{r}),\\ \mathbf{b}(\mathbf{k},\mathbf{r})&=\mathbf{b}^i(\mathbf{k},\mathbf{r})+\mathbf{b}^s(\mathbf{k},\mathbf{r}),\end{aligned} \tag{3}$$

It is well known that the above terms for the fields can be expanded into the sum of spherical harmonic functions as follows when the spherical coordinate is composed of r, θ, ϕ, where $\mathbf{r}=r(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. Here the amplitude of the incident electric field is normalized [6],

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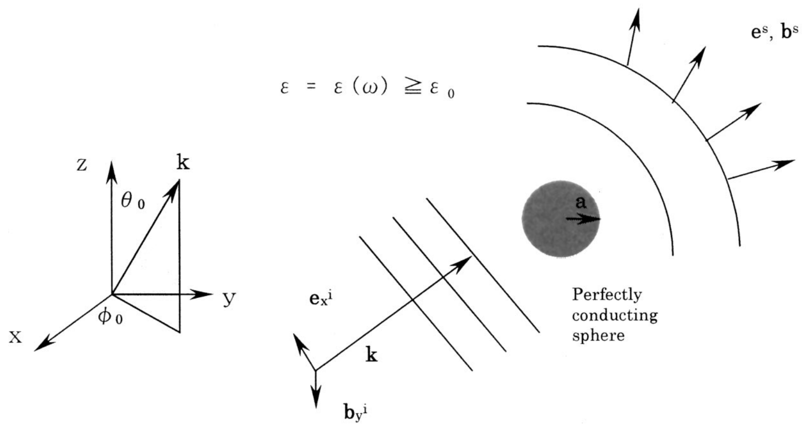

FIG. 1. Electromagnetic field scattered by a perfectly conducting sphere.

$$e_r^i(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}\sum_{n=1}^{\infty}(-i)^n(2n+1)\frac{j_n(kr)}{ikr}\times\left[2\sum_{m=1}^{n}\frac{(n-m)!m}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\times P_n^m(\cos\theta)\sin m(\phi-\phi_0)\right], \tag{4}$$

$$e_\theta(\mathbf{k},\mathbf{r})=e_\theta^i(\mathbf{k},\mathbf{r})+e_\theta^s(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}F[(A-MC)Y_1+(ND-B)Y_2], \tag{5}$$

$$e_\phi(\mathbf{k},\mathbf{r})=e_\phi^i(\mathbf{k},\mathbf{r})+e_\phi^s(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}F[(A-MC)Y_3+(ND-B)Y_4], \tag{6}$$

$$b_r^i(\mathbf{k},\mathbf{r})=-\frac{1}{\sqrt{(2\pi)^3}}\sum_{n=1}^{\infty}(-i)^n(2n+1)\frac{k}{\omega}\frac{j_n(kr)}{ikr}\times\left[\frac{dP_n(\cos\theta_0)}{d\theta_0}P_n(\cos\theta)+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\times P_n^m(\cos\theta)\cos m(\phi-\phi_0)\right], \tag{7}$$

$$b_\theta(\mathbf{k},\mathbf{r})=b_\theta^i(\mathbf{k},\mathbf{r})+b_\theta^s(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}\frac{k}{\omega}F[(B-MD)Y_3+(NC-A)Y_4], \tag{8}$$

$$b_\phi(\mathbf{k},\mathbf{r})=b_\phi^i(\mathbf{k},\mathbf{r})+b_\phi^s(\mathbf{k},\mathbf{r})=-\frac{1}{\sqrt{(2\pi)^3}}\frac{k}{\omega}F[(B-MD)Y_1+(NC-A)Y_2], \tag{9}$$

where $1/\sqrt{(2\pi)^3}$ is a factor introduced for calculating convenience, *A, B, C, D,* and *F* are expressed as

$$A=\frac{\frac{d}{dr}[rj_n(kr)]}{ikr},\quad B=j_n(kr),\qquad C=\frac{\frac{d}{dr}[rh_n^{(2)}(kr)]}{ikr},\quad D=h_n^{(2)}(kr), \tag{10}$$

$$F=\sum_{n=1}^{\infty}(-i)^n\frac{(2n+1)}{n(n+1)}, \tag{11}$$

coefficients *M* and *N* are determined in order to satisfy the boundary condition of the electric field on the surface of the perfectly conducting sphere $\mathbf{e}^i|_{r=a}+\mathbf{e}^s|_{r=a}=0$,

$$M=\left.\frac{\frac{d}{dr}[rj_n(kr)]}{\frac{d}{dr}[rh_n^{(2)}(kr)]}\right|_{r=a},\quad N=\left.\frac{j_n(kr)}{h_n^{(2)}(kr)}\right|_{r=a}, \tag{12}$$

and

$$Y_1=2\sum_{m=1}^{n}\frac{(n-m)!m}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{dP_n^m(\cos\theta)}{d\theta}\times\sin m(\phi-\phi_0), \tag{13}$$

$$Y_2=2\sum_{m=1}^{n}\frac{(n-m)!m}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{P_n^m(\cos\theta)}{\sin\theta}\times\sin m(\phi-\phi_0), \tag{14}$$

$$Y_3=2\sum_{m=1}^{n}\frac{(n-m)!m^2}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta)}{\sin\theta}\times\cos m(\phi-\phi_0), \quad (15)$$

$$Y_4=\frac{dP_n(\cos\theta_0)}{d\theta_0}\frac{dP_n(\cos\theta)}{d\theta}+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta)}{d\theta}\times\cos m(\phi-\phi_0). \quad (16)$$

Here $j_n(kr)$ are the spherical Bessel functions, $h_n^{(2)}(kr)$ are the spherical Hankel functions, $P_n(\cos\theta)$ are the Legendre functions, and $P_n^m(\cos\theta)$ are the associated Legendre functions, where m and n are integers, i.e., $m=1,2,3,...,n$ and $n=1,2,3,...$. The near field around the sphere is represented by the field component having the higher order r dependence r^{-n}, $n=2,3,...$.

III. ORTHOGONALITY RELATION

Using the electromagnetic field functions given by Eqs. (4)–(9), we first calculate the quantity S_r,

$$S_r=\frac{1}{\mu_0}[(e'_\theta\times b^*_\phi-e'_\phi\times b^*_\theta)+(e^*_\theta\times b'_\phi-e^*_\phi\times b'_\theta)], \quad (17)$$

where S_r is the radial component of the vector **S**,

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{e}'\times\mathbf{b}^*+\mathbf{e}^*\times\mathbf{b}'). \quad (18)$$

Here $\mathbf{e}'$ (or $\mathbf{b}'$) is the electric (or magnetic) field given also by Eqs. (4)–(9) with wave vector $\mathbf{k}'$ and frequency ω', and $*$ denotes the complex conjugate. Then, using Eqs. (4)–(9) and (17), we obtain the following equations (see Appendixes A and B):

$$\frac{\int\nabla\cdot\mathbf{S}\,d\mathbf{r}}{i(\omega-\omega')}=\lim_{r\to\infty}\frac{\int_0^{2\pi}d\phi\int_0^{\pi}S_r r^2\sin\theta d\theta}{i(\omega-\omega')}$$

$$=\frac{1}{4\pi k^2}\left\{\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[(1+2|M|^2-M-M^*)2\sum_{m=1}^{n}\frac{(n-m)!m^2}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}\cos m(\phi_0-\phi_0')\right.\right.$$
$$+(1+2|N|^2-N-N^*)\left(P_n^1(\cos\theta_0)P_n^1(\cos\theta_0')\right.$$
$$\left.\left.\left.+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\right)\cos m(\phi_0-\phi_0')\right]\right\}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(k-k') \quad (19)$$

$$=\frac{1}{4\pi k^2}\left\{\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[P_n^1(\cos\theta_0)P_n^1(\cos\theta_0')\right.\right.$$
$$\left.\left.+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\left(m^2\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}+\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\right)\cos m(\phi_0-\phi_0')\right]\right\}$$
$$\times\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(k-k') \quad (20)$$

$$=\frac{1}{k^2\sin\theta_0}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(k-k')\,\delta(\theta_0-\theta_0')\,\delta(\phi_0-\phi_0'). \quad (21)$$

On the other hand, starting from Maxwell's equations,

$$\nabla\times\mathbf{e}=i\omega\mathbf{b},\quad \frac{1}{\mu_0}\nabla\times\mathbf{b}=-i\omega\varepsilon\mathbf{e}, \quad (22)$$

we obtain

$$\frac{\nabla\cdot\mathbf{S}}{i(\omega-\omega')}=\frac{\omega\varepsilon-\omega'\varepsilon'}{\omega-\omega'}\mathbf{e}'\cdot\mathbf{e}^*+\frac{1}{\mu_0}\mathbf{b}'\cdot\mathbf{b}^*. \quad (23)$$

The combination of Eqs. (21) and (23) yields an orthogonality relation of the electromagnetic field consisting of incident and scattering fields,

$$\int\left(\frac{\omega\varepsilon-\omega'\varepsilon'}{\omega-\omega'}\mathbf{e}'\cdot\mathbf{e}^{*}+\frac{1}{\mu_0}\mathbf{b}'\cdot\mathbf{b}^{*}\right)d\mathbf{r}$$
$$=\frac{\int\nabla\cdot\mathbf{S}d\mathbf{r}}{i(\omega-\omega')}$$
$$=\frac{1}{k^2\sin\theta_0}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\delta(k-k')\delta(\theta_0-\theta_0')\delta(\phi_0-\phi_0')$$
$$=\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\delta(\mathbf{k}-\mathbf{k}'). \quad (24)$$

IV. EXPANSION OF THE FIELD ENERGY

Let us assume that the electromagnetic field is given as a superposition of the fields with various wave vectors $\mathbf{k}$,

$$\mathbf{E}(\mathbf{r},t)=\int_{-\infty}^{\infty}d\mathbf{k}\,C_{\mathbf{k}}\mathbf{e}(\mathbf{k},\mathbf{r})e^{-i\omega(\mathbf{k})t},$$
$$\mathbf{B}(\mathbf{r},t)=\int_{-\infty}^{\infty}d\mathbf{k}\,C_{\mathbf{k}}\mathbf{b}(\mathbf{k},\mathbf{r})e^{-i\omega(\mathbf{k})t}. \quad (25)$$

Here $C_{\mathbf{k}}$ denotes complex amplitudes, and $\mathbf{e}(\mathbf{k},\mathbf{r})$ and $\mathbf{b}(\mathbf{k},\mathbf{r})$ are defined by Eqs. (4)–(9). In order to assure the real property of the field, $\mathbf{E}(\mathbf{r},t)=\mathbf{E}^{*}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)=\mathbf{B}^{*}(\mathbf{r},t)$, we introduce the definitions

$$C_{-\mathbf{k}}\equiv C_{\mathbf{k}}^{*},\quad \omega(-\mathbf{k})\equiv-\omega(\mathbf{k}), \quad (26)$$

and apply the equations

$$\mathbf{e}(-\mathbf{k},\mathbf{r})=\mathbf{e}^{*}(\mathbf{k},\mathbf{r}),\quad \mathbf{b}(-\mathbf{k},\mathbf{r})=\mathbf{b}^{*}(\mathbf{k},\mathbf{r}), \quad (27)$$

which are shown by the analytic continuation of the fields given by Eqs. (4)–(9) with the wave vector $\mathbf{k}$ into those with the wave vector $-\mathbf{k}$ [7].

The electric field energy W_E stored in the medium is expressed as

$$\frac{\partial}{\partial t}W_E\equiv\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},t)\frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t}d\mathbf{r}, \quad (28)$$

where $\mathbf{D}(\mathbf{r},t)$ is the electric field displacement given by

$$\mathbf{D}(\mathbf{r},t)=\int_{-\infty}^{\infty}d\mathbf{k}\,\varepsilon(\omega)\mathbf{e}(\mathbf{k},\mathbf{r})C_{\mathbf{k}}e^{-i\omega(\mathbf{k})t}. \quad (29)$$

Using Eqs. (25) and (29), we obtain (see Appendix C),

$$W_E=\frac{1}{2}\int C_{\mathbf{k}}'C_{\mathbf{k}}^{*}e^{i(\omega-\omega')t}d\mathbf{k}\,d\mathbf{k}'\int\frac{\omega\varepsilon-\omega'\varepsilon'}{\omega-\omega'}\mathbf{e}'\cdot\mathbf{e}^{*}d\mathbf{r}. \quad (30)$$

In addition, the magnetic energy W_B stored in the whole space is expressed as

$$W_B=\frac{1}{2}\int C_{\mathbf{k}}'C_{\mathbf{k}}^{*}e^{i(\omega-\omega')t}d\mathbf{k}\,d\mathbf{k}'\int\frac{1}{\mu_0}\mathbf{b}'\cdot\mathbf{b}^{*}d\mathbf{r}. \quad (31)$$

Therefore, the total field energy W stored in the dispersive medium is the sum of W_E and W_B,

$$W=W_E+W_B=\frac{1}{2}\int C_{\mathbf{k}}'C_{\mathbf{k}}^{*}e^{i(\omega-\omega')t}d\mathbf{k}\,d\mathbf{k}'$$
$$\times\int\left(\frac{\omega\varepsilon-\omega'\varepsilon'}{\omega-\omega'}\mathbf{e}'\cdot\mathbf{e}^{*}+\frac{1}{\mu_0}\mathbf{b}'\cdot\mathbf{b}^{*}\right)d\mathbf{r}. \quad (32)$$

Applying the derived orthogonal relation given by Eq. (24) to Eq. (32), the total field energy can be rewritten as

$$W=\frac{1}{2}\int C_{\mathbf{k}}C_{\mathbf{k}}^{*}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}d\mathbf{k}. \quad (33)$$

Furthermore, if we introduce normalized amplitudes $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{*}$ as

$$a_{\mathbf{k}}=\sqrt{\frac{1}{2\hbar\omega}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}}C_{\mathbf{k}},\quad a_{\mathbf{k}}^{*}=\sqrt{\frac{1}{2\hbar\omega}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}}C_{\mathbf{k}}^{*}, \quad (34)$$

then the total field energy of the electromagnetic field consisting of incident and scattering fields can be expressed as

$$W=\int\hbar\omega(\mathbf{k})a_{\mathbf{k}}a_{\mathbf{k}}^{*}d\mathbf{k}. \quad (35)$$

Equation (35) shows that the total field is given by the sum of the energies of independent harmonic oscillators.

V. CONCLUSION

Considering a typical three-dimensional scattering electromagnetic field in a dispersive medium, we derived an orthogonality relation between fields with different wave vectors. Here the doublet of incident and scattering fields comprised an orthonormal set. Expanding the electromagnetic field energy stored in the whole space using the derived orthogonality relation, we showed that the total field energy was expressed as the sum of the energies of independent harmonic oscillators. The above result will be useful for studying the quantal interaction between a scattering electromagnetic field and an atom.

APPENDIX A: AN EXPRESSION OF THE δ FUNCTIONS IN SPHERICAL COORDINATES

Let us derive an expression of the δ function in spherical coordinates. Under the Cartesian coordinates, a plane electromagnetic field in a dispersive dielectric can be written as

$$\mathbf{e}=\frac{1}{\sqrt{(2\pi)^3}}\hat{\mathbf{e}}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega(\mathbf{k})t},\quad \mathbf{b}=\frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\omega}\mathbf{k}\times\hat{\mathbf{e}}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega(\mathbf{k})t}, \quad (A1)$$

where $\hat{\mathbf{e}}$ is the unit vector, $\mathbf{k}=(k_x,k_y,k_z)$ is the wave vector which is a function of angular frequency ω, and its absolute square is given by

$$k^2=|\mathbf{k}|^2=\omega^2\varepsilon\mu_0. \quad (A2)$$

Using the equation

$$\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}d\mathbf{r}=2\pi\delta(\mathbf{k}-\mathbf{k}'), \tag{A3}$$

we obtain the following equation:

$$\frac{\int\nabla\cdot\mathbf{S}\,d\mathbf{r}}{i(\omega-\omega')}=\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(k_x-k_x')\,\delta(k_y-k_y')\,\delta(k_z-k_z')$$
$$=\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(\mathbf{k}-\mathbf{k}'), \tag{A4}$$

where $\mathbf{S}$ is defined by Eq. (18), and $\mathbf{e}'$ and $\mathbf{b}'$ are defined by Eq. (A1) with the wave vector $\mathbf{k}'=(k_x',k_y',k_z')$.

Under spherical coordinates, Eq. (A4) can be transformed into

$$\frac{\int\nabla\cdot\mathbf{S}\,d\mathbf{r}}{i(\omega-\omega')}=\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\frac{1}{k^2\sin\theta_0}\,\delta(k-k')\,\delta(\theta_0-\theta_0')\,\delta(\phi_0-\phi_0'), \tag{A5}$$

where the wave vectors $\mathbf{k}$ and $\mathbf{k}'$ are expressed as $\mathbf{k}=k(\sin\theta_0\cos\phi_0,\sin\theta_0\sin\phi_0,\cos\theta_0)$ and $\mathbf{k}'=k'(\sin\theta_0'\cos\phi_0',\sin\theta_0'\sin\phi_0',\cos\theta_0')$, where $\cos\theta_0$, $\cos\phi_0$, $\cos\theta_0'$, and $\cos\phi_0'$ are the direction cosines of $\mathbf{k}$ and $\mathbf{k}'$, respectively.

On the other hand, the plane electromagnetic field given by Eq. (A1) can be expanded into the sum of the spherical harmonic functions similar to Eqs. (4)–(9),

$$e_r(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}\sum_{n=1}^{\infty}(-i)^n(2n+1)\frac{j_n(kr)}{ikr}\times\left[2\sum_{m=1}^{n}\frac{(n-m)!m}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}P_n^m(\cos\theta)\times\sin m(\phi-\phi_0)\right], \tag{A6}$$

$$e_\theta(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}F(AY_1-BY_2), \tag{A7}$$

$$e_\phi(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}F(AY_3-BY_4), \tag{A8}$$

$$b_r(\mathbf{k},\mathbf{r})=-\frac{1}{\sqrt{(2\pi)^3}}\sum_{n=1}^{\infty}(-i)^n(2n+1)\frac{k}{\omega}\frac{j_n(kr)}{ikr}\times\left[\frac{dP_n(\cos\theta_0)}{d\theta_0}P_n(\cos\theta)+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0}P_n^m(\cos\theta)\times\cos m(\phi-\phi_0)\right], \tag{A9}$$

$$b_\theta(\mathbf{k},\mathbf{r})=\frac{1}{\sqrt{(2\pi)^3}}\frac{k}{\omega}F(BY_3-AY_4), \tag{A10}$$

$$b_\phi(\mathbf{k},\mathbf{r})=-\frac{1}{\sqrt{(2\pi)^3}}\frac{k}{\omega}F(BY_1-AY_2). \tag{A11}$$

Substituting A, B, F, Y_1, Y_2, Y_3, and Y_4 defined by Eqs. (10)–(16) into Eq. (A4) and using Eq. (17), we have

$$\frac{\int\nabla\cdot\mathbf{S}\,d\mathbf{r}}{i(\omega-\omega')}=\lim_{r\to\infty}\frac{\int_0^{2\pi}d\phi\int_0^{\pi}S_r r^2\sin\theta d\theta}{i(\omega-\omega')}$$
$$=\frac{1}{k^2}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(k-k')\,\frac{1}{4\pi}\left\{\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[P_n^1(\cos\theta_0)P_n^1(\cos\theta_0')+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\left(m^2\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}+\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\right)\cos m(\phi_0-\phi_0')\right]\right\}. \tag{A12}$$

The results of Eqs. (A5) and (A12) must be the same, so that we obtain an expression of the δ function as

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\Bigg[P_n^1(\cos\theta_0)P_n^1(\cos\theta_0')$$
$$+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\left(m^2\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}+\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\right)\cos m(\phi_0-\phi_0')\Bigg]$$
$$=\frac{4\pi}{\sin\theta_0}\delta(\theta_0-\theta_0')\delta(\phi_0-\phi_0'). \tag{A13}$$

APPENDIX B: THE CALCULATION OF $\nabla\cdot\mathbf{S}$

The quantity S_r defined by Eq. (17) can be obtained from Eqs. (4)–(9) as

$$S_r=\frac{1}{\mu_0}[(e_\theta'\times b_\phi^*-e_\phi'\times b_\theta^*)+(e_\theta^*\times b_\phi'-e_\phi^*\times b_\theta')] \tag{B1}$$

$$=-F'F^*\Bigg\{\left(\frac{k}{\omega\mu_0}(A'-M'C')(B^*-M^*D^*)+\frac{k'}{\omega'\mu_0}(A^*-M^*C^*)(B'-M'D')\right)(Y_1'Y_1+Y_3'Y_3)$$
$$+\left(\frac{k}{\omega\mu_0}(N'D'-B')(N^*C^*-A^*)+\frac{k'}{\omega'\mu_0}(N^*D^*-B^*)(N'C'-A')\right)(Y_2'Y_2+Y_4'Y_4)$$
$$+\left(\frac{k}{\omega\mu_0}(A'-M'C')(N^*C^*-A^*)+\frac{k'}{\omega'\mu_0}(N^*D^*-B^*)(B'-M'D')\right)(Y_1'Y_2+Y_3'Y_4)$$
$$+\left(\frac{k}{\omega\mu_0}(N'D'-B')(B^*-M^*D^*)+\frac{k'}{\omega'\mu_0}(A^*-M^*C^*)(N'C'-A')\right)(Y_2'Y_1+Y_4'Y_3)\Bigg\}, \tag{B2}$$

where A', B', C', D', F', M', N', Y_1', Y_2', Y_3' and Y_4' are given by replacing k, θ_0, ϕ_0, n, and m in Eqs. (11) and (13)–(16) with k', θ_0', ϕ_0', n', and m'.

By using the orthogonality relations,

$$\int_0^{2\pi}\sin m'(\phi-\phi_0')\sin m(\phi-\phi_0)d\phi=\int_0^{2\pi}\cos m'(\phi-\phi_0')\cos m(\phi-\phi_0)d\phi=\begin{cases}\pi\cos m(\phi_0-\phi_0'), & m=m'\\ 0, & m\neq m',\end{cases} \tag{B3}$$

$$\int_0^{\pi}\left(\frac{dP_n^m(\cos\theta)}{d\theta}\frac{dP_{n'}^m(\cos\theta)}{d\theta}+m^2\frac{P_n^m(\cos\theta)}{\sin\theta}\frac{P_{n'}^m(\cos\theta)}{\sin\theta}\right)\sin\theta d\theta=\begin{cases}\dfrac{2(n+m)!n(n+1)}{(n-m)!(2n+1)}, & n=n'\\ 0, & n\neq n',\end{cases} \tag{B4}$$

and the identity

$$\int_0^{\pi}\left(P_{n'}^m(\cos\theta)\frac{dP_n^m(\cos\theta)}{d\theta}+P_n^m(\cos\theta)\frac{dP_{n'}^m(\cos\theta)}{d\theta}\right)d\theta=0, \tag{B5}$$

we obtain the integral of Sr over a spherical surface;

$$\int_0^{2\pi}d\phi\int_0^{\pi}S_r\sin\theta d\theta=-\sum_{n=1}^{\infty}\frac{4\pi(2n+1)}{n(n+1)}\Bigg\{\left(\frac{k}{\omega\mu_0}(A'-M'C')(B^*-M^*D^*)+\frac{k'}{\omega'\mu_0}(A^*-M^*C^*)(B'-M'D')\right)$$
$$\times 2\sum_{m=1}^{n}\frac{(n-m)!m^2}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}\cos m(\phi_0'-\phi_0)$$
$$+\left(\frac{k}{\omega\mu_0}(N'D'-B')(N^*C^*-A^*)+\frac{k'}{\omega'\mu_0}(N^*D^*-B^*)(N'C'-A')\right)$$
$$\times\left(\frac{dP_n(\cos\theta_0)}{d\theta_0}\frac{dP_n(\cos\theta_0')}{d\theta_0'}+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\,\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\cos m(\phi_0'-\phi_0)\right)\Bigg\}. \tag{B6}$$

Considering further the following equations:

$$\lim_{r\to\infty} A = \lim_{r\to\infty} \frac{\frac{d}{dr}[rj_n(kr)]}{ikr} = \lim_{r\to\infty} i\,\frac{\sin\left(kr - \frac{n+1}{2}\pi\right)}{kr}, \tag{B7}$$

$$\lim_{r\to\infty} B = \lim_{r\to\infty} j_n(kr) = \lim_{r\to\infty} \frac{\cos\left(kr - \frac{n+1}{2}\pi\right)}{kr}, \tag{B8}$$

$$\lim_{r\to\infty} C = \lim_{r\to\infty} \frac{\frac{d}{dr}[rh_n^{(2)}(kr)]}{ikr} = -\lim_{r\to\infty} i^{n+1}\frac{e^{-ikr}}{kr}, \tag{B9}$$

$$\lim_{r\to\infty} D = \lim_{r\to\infty} h_n^{(2)}(kr)) = \lim_{r\to\infty} i^{n+1}\frac{e^{-ikr}}{kr}, \tag{B10}$$

$$\lim_{r\to\infty} \frac{\sin(k-k')r}{k-k'} = \pi\delta(k-k'), \quad \lim_{r\to\infty} \frac{e^{i(k-k')r}}{i(k-k')} = \pi\delta(k-k'), \tag{B11}$$

we obtain from Eq. (B6),

$$\begin{aligned}\frac{\int \nabla\cdot\mathbf{S}\,d\mathbf{r}}{i(\omega-\omega')} &= \lim_{r\to\infty}\frac{\int_0^{2\pi}d\phi\int_0^{\pi}S_r r^2\sin\theta\,d\theta}{i(\omega-\omega')}\\ &= \frac{1}{4\pi k^2}\Bigg\{\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\Bigg[(1+2|M|^2-M-M^*)2\sum_{m=1}^{n}\frac{(n-m)!m^2}{(n+m)!}\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}\cos m(\phi_0-\phi_0')\\ &\quad+(1+2|N|^2-N-N^*)\left(P_n^1(\cos\theta_0)P_n^1(\cos\theta_0')+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\right)\cos m(\phi_0-\phi_0')\Bigg]\Bigg\}\\ &\quad\times\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\,\delta(k-k').\end{aligned} \tag{B12}$$

Applying the equation

$$h_n^{(2)}(kr)+h_n^{(2)*}(kr)=2j_n(kr) \tag{B13}$$

and substituting the equations from Eq. (12),

$$2|M|^2-M-M^* = \left[\frac{2\left[\frac{d}{dr}[rj_n(kr)]\right]^2}{\frac{d}{dr}[rh_n^{(2)}(kr)]\frac{d}{dr}[rh_n^{(2)*}(kr)]} - \frac{\frac{d}{dr}[rj_n(kr)]}{\frac{d}{dr}[rh_n^{(2)}(kr)]} - \frac{\frac{d}{dr}[rj_n(kr)]}{\frac{d}{dr}[rh_n^{(2)*}(kr)]}\right]_{r=a} = 0, \tag{B14}$$

$$2|N|^2-N-N^* = \left[\frac{2[j_n(kr)]^2}{h_n^{(2)}(kr)h_n^{(2)*}(kr)} - \frac{j_n(kr)}{h_n^{(2)}(kr)} - \frac{j_n(kr)}{h_n^{(2)*}(kr)}\right]_{r=a} = 0, \tag{B15}$$

into Eq. (B12), Eq. (B12) can be rewritten as

$$\begin{aligned}\frac{\int \nabla\cdot\mathbf{S}\,d\mathbf{r}}{i(\omega-\omega')} &= \frac{1}{4\pi k^2}\frac{\partial\omega^2\varepsilon}{\omega\partial\omega}\delta(k-k')\Bigg\{\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\Bigg[P_n^1(\cos\theta_0)P_n^1(\cos\theta_0')\\ &\quad+2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\left(m^2\frac{P_n^m(\cos\theta_0)}{\sin\theta_0}\frac{P_n^m(\cos\theta_0')}{\sin\theta_0'}+\frac{dP_n^m(\cos\theta_0)}{d\theta_0}\frac{dP_n^m(\cos\theta_0')}{d\theta_0'}\right)\cos m(\phi_0-\phi_0')\Bigg]\Bigg\}.\end{aligned} \tag{B16}$$

According to Eq. (A13) derived in Appendix A, the final result shown by Eq. (21) is obtained,

$$\frac{\int \nabla \cdot \mathbf{S}\, d\mathbf{r}}{i(\omega-\omega')} = \frac{1}{k^2 \sin\theta_0} \frac{\partial \omega^2 \varepsilon}{\omega \partial \omega} \delta(k-k')\,\delta(\theta_0-\theta_0')\,\delta(\phi_0-\phi_0'). \tag{B17}$$

APPENDIX C: THE ELECTRIC ENERGY IN A DISPERSIVE MEDIUM

From Eq. (29), the following equation can be obtained:

$$\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} = \int_{-\infty}^{\infty} d\mathbf{k}\, C_{\mathbf{k}} e^{-i\omega(\mathbf{k})t}(-i\omega\varepsilon)\mathbf{e}(\mathbf{k},\mathbf{r}). \tag{C1}$$

Using Eqs. (25) and (C1), and $\varepsilon(-\mathbf{k}) = \varepsilon(\mathbf{k}) = \varepsilon^*(\mathbf{k})$ because of the lossless effect of the medium, we obtain

$$\begin{aligned} \mathbf{E}(\mathbf{r},t)\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} &= \frac{1}{2}\left(\mathbf{E}'(\mathbf{r},t)\frac{\partial \mathbf{D}^*(\mathbf{r},t)}{\partial t} + \mathbf{E}^*(\mathbf{r},t)\frac{\partial \mathbf{D}'(\mathbf{r},t)}{\partial t}\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} d\mathbf{k}\, d\mathbf{k}'\, C_{\mathbf{k}}^* C_{\mathbf{k}'} e^{-i[\omega(\mathbf{k}')-\omega(\mathbf{k})]t}(i\omega\varepsilon - i\omega'\varepsilon')\mathbf{e}^*(\mathbf{k},\mathbf{r})\cdot\mathbf{e}(\mathbf{k}',\mathbf{r}) \end{aligned} \tag{C2}$$

and therefore, Eq. (30) can be derived as

$$\begin{aligned} W_E &= \int d\mathbf{r} \int_{-\infty}^{t} \mathbf{E}(\mathbf{r},t)\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} d\mathbf{k}\, d\mathbf{k}'\, C_{\mathbf{k}}^* C_{\mathbf{k}'} e^{-i[\omega(\mathbf{k}')-\omega(\mathbf{k})]t} \int \frac{\omega\varepsilon - \omega'\varepsilon'}{\omega-\omega'} \mathbf{e}^*(\mathbf{k},\mathbf{r})\cdot\mathbf{e}(\mathbf{k}',\mathbf{r})d\mathbf{r}, \end{aligned} \tag{C3}$$

where the field energy density is assumed to vanish at $t = -\infty$.

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